Introduction to Segal CFT, with connections to VOAs and conformal nets

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(partly joint work with André Henriques)

Overview

- In the last talk we saw two different ways of axiomatizing the vacuum representation of a unitary 2d chiral CFT: conformal nets and unitary VOAs
- In this talk we will discuss a third axiomatization, Segal (functorial) CFTs.
- We'll see how Segal CFTs compare to conformal nets and VOAs, as well as how Henriques' partially thin surfaces can be used to simultaneously encode all three frameworks.

Part 1: The vacuum sector

Functorial field theories

The first version of the Atiyah-Segal axioms describe a field theory as a symmetric monoidal functor

$$g\!-\!\mathrm{Bord}_d \to \text{“Vec”}$$

A full conformal field theory correspond to complex cobordisms with $d = 2$ (i.e. Riemann surfaces), but that structure does not capture most chiral CFTs.

However the “local” version of these axioms corresponds to the vacuum sector for a chiral CFT.

Segal CFTs in the vacuum sector (preliminary version)

Data of a vacuum Segal CFT:

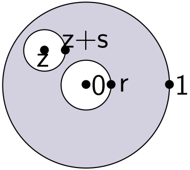
- A Hilbert space H_0
- For every n-to-1 genus zero Riemann surface Σ with boundary components parametrized by S^1, a map $Y_\Sigma : \bigotimes_{\partial_{in}\Sigma} H_0 \to H_0$

Such that:

- Gluing of surfaces $\longleftrightarrow$ composition of maps (up to scalar)
- Y_Σ is holomorphic in Σ

Vacuum Segal CFTs vs VOAs

The dictionary between vacuum Segal CFT and VOAs is:



$$= \Sigma \quad \longmapsto \quad Y_\Sigma(v \otimes u) = Y(s^{L_0} v, z) r^{L_0} u$$

- Huang has constructed genus zero CFTs from vertex operator algebras, with the Hilbert space replaced by a purpose-built topological completion. He also shows that a version of this definition with punctures is equivalent to VOAs.

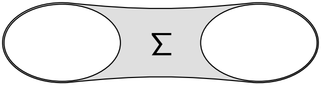
- Showing that these maps are continuous for unitary VOAs is open.

- Positive solution for WZW models, subtheories, etc. [T '17]. Conjecturally holds for all unitary VOAs.

Enhanced vacuum Segal CFTs

Joint work with Henriques:

- An *enhanced* vacuum Segal CFT allows Σ to be partially thin in the sense of Henriques.

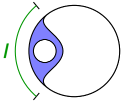


- As "thick" surfaces are dense, this is an analytic condition not requiring extra data.
- Conjecture: every vacuum Segal CFT (and every unitary VOA) extends to an enhanced vacuum Segal CFT.
- This is a strengthening of the conjecture that every VOA integrates to a conformal net.

Enhanced vacuum Segal CFTs and conformal nets

- We've seen how vacuum Segal CFTs contain the data of a VOA
- The enhanced version also contain the data of a conformal net

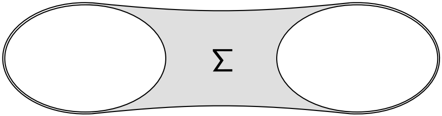
$$\mathcal{A}(I) = vNA\Bigg(\Bigg\{ Y_\Sigma(v \otimes \cdot) \;\Bigg|\; \Sigma =$$



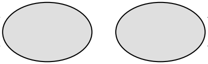
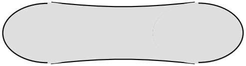
$$\Bigg\}\Bigg)$$

- Every conformal net can be constructed from such local insertions, but it a conjecture that every conformal net generates an enhanced vacuum Segal CFT.

Enhanced vacuum Segal CFT from conformal nets



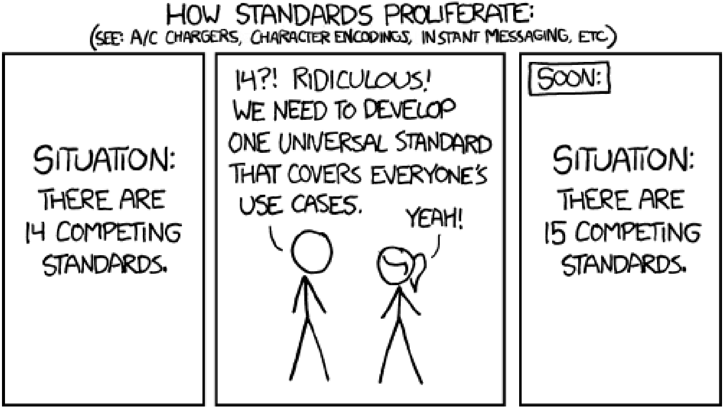
Densely define $Y_\Sigma : H_{vacuum} \otimes H_{vacuum} \to H_{vacuum}$ by

- $Y_\Sigma($$) =$ 
- Y_Σ intertwines action of $\mathcal{A}\Big($$\Big)$

Conjecture that Y_Σ is continuous for any conformal net. Verified in most known examples with Henriques.

If so, surfaces like Σ should generate an enhanced vacuum Segal CFT.

Enhanced vacuum Segal CFTs can be studied both as conformal nets and VOAs, hopefully avoiding:



(Source: Randall Munroe, https://xkcd.com/927/)

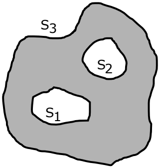
Part 2: Representations and higher genus

Weakly conformal field theories

- Chiral CFTs provide examples of what Segal calls "weakly conformal field theories" and I'll call "Segal CFTs." This is a more general notion than the prototype Atiyah-Segal axioms we saw.
- Unlike VOAs/conformal nets/vacuum Segal CFTs, the axioms of Segal CFT are not 'minimal.' The data of a Segal CFT includes all of the representations, all of the categorical information, and more.

Example: WZW models

- $\mathfrak{g}$ compact simple Lie algebra, $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g}_{\mathbb{C}})$.
- We describe Segal's proposed tensor structure on $\mathcal{C} := \mathrm{Rep}_k(L\mathfrak{g})$, the category of positive energy representations of $L\mathfrak{g}$.
- There is a "tensor product" for every complex pair of pants with parametrized boundary.

$\Sigma =$ 

$\longleftrightarrow - \boxtimes_\Sigma - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

Warmup: tensor product of Lie algebra modules

If V, W are $\mathfrak{g}$-modules, the tensor product $V \otimes W$ is a $\mathfrak{g}$-module:

- equipped with a bilinear map $Z : V \times W \to V \otimes W$, such that

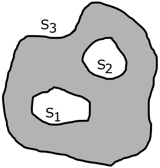
$$X \cdot Z(v, w) = Z(X \cdot v, w) + Z(v, X \cdot w)$$

  for all $X \in \mathfrak{g}$
- which is universal, so that if U is a $\mathfrak{g}$-module equipped with $Y : V \times W \to U$, then Y factors through Z.

$$\begin{tikzcd} V \times W \arrow[r, "Z"] \arrow[dr, "Y"'] & V \otimes W \arrow[d, dashed, "\exists!\ \mathfrak{g}\text{-map}"] \\ & U \end{tikzcd}$$

Existence of such a module is shown by explicit construction.

Segal's holomorphic induction

$\Sigma =$ 

$\longleftrightarrow - \boxtimes_\Sigma - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

If $V_1, V_2 \in \mathcal{C} = \mathrm{Rep}_k(L\mathfrak{g})$, then $V_1 \boxtimes_\Sigma V_2$ is a $L\mathfrak{g}$-module:

- equipped with a bilinear map $Z_\Sigma : V_1 \times V_2 \to V_1 \boxtimes_\Sigma V_2$ satisfying the *Segal commutation relations:*

  $$f|_{S_3} \cdot Z_\Sigma(v, w) = Z_\Sigma(f|_{S_1} \cdot v, w) + Z_\Sigma(v, f|_{S_2} \cdot w)$$

  for all $f \in \mathcal{O}_{hol}(\Sigma; \mathfrak{g}_\mathbb{C})$

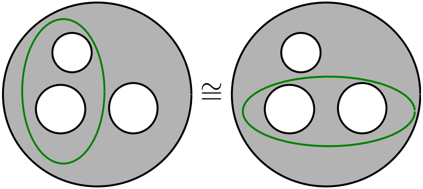
- which is universal

Problems: Existence? Positivity energy? Associativity?

Associativity

The associativity property says that we should have an equivalence between the follows functors

$$\left(\mathcal{C}\times\mathcal{C}\right)\times\mathcal{C}\to\mathcal{C}\cong\mathcal{C}\times\left(\mathcal{C}\times\mathcal{C}\right)\to\mathcal{C}$$



We also want the maps Z_Σ to be compatible with these isomorphisms.

- We can also try to do holomorphic induction for any complex cobordism Σ with $\partial_{out}\Sigma \neq \emptyset$
- The unit disk D is assigned a representation V_D equipped with a map $Z_D : \mathbb{C} \to V_D$ satisfying

  $$f \cdot Z_D = 0$$

  for all $f \in \mathcal{O}_{hol}(D; \mathfrak{g}_{\mathbb{C}})$.

  i.e. V_D is the vacuum sector, and $Z_D(1) = \Omega_D$ is the vacuum vector.

Unitary chiral Segal CFT

1) For every smooth, oriented 1-manifold S,
   a) a Hilb-linear category $\mathcal{C}(S)$,
   b) equipped with a functor $H_\bullet : \mathcal{C}(S) \to \mathrm{Hilb}$.

   $\varphi : S_1 \cong S_2$ induces $\mathcal{C}(S_1) \cong \mathcal{C}(S_2)$, Comparison functor $\mathcal{C}(S_1) \times \mathcal{C}(S_2) \to \mathcal{C}(S_1 \sqcup S_2)$, etc.

2) For every complex cobordism Σ with $\partial_{out}\Sigma \neq \emptyset$
   a) a functor $F_\Sigma : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma)$
   b) a natural transformation $Z_\Sigma : H_\lambda \to H_{F_\Sigma(\lambda)}$ $(\lambda \in \mathcal{C}(\partial_{in}\Sigma))$

   Compatible with composition (projectively, unless anomaly is addressed).

3) For every path of surfaces $[0,1] \ni t \mapsto \Sigma_t$
   a) an equivalence $F_{\Sigma_0} \cong F_{\Sigma_1}$
   b) "chirality", "projectively flat"

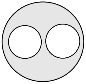
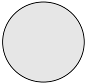
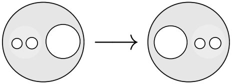
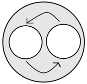
Rational CFTs: $\mathcal{C}(S)$ is finitely semisimple, and $\partial_{out} = \emptyset$ is allowed.

Part (a) corresponds to a modular functor, Part (b) is the CFT.

Tensor category structure

Segal CFT answers the question "What is the structure of the representation theory of a chiral CFT?"

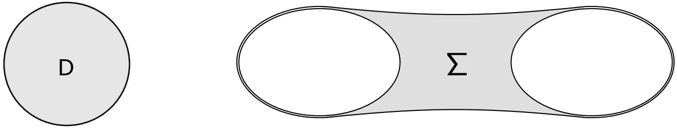
The (a) part makes $\mathcal{C}(S^1)$ into a braided tensor category.

- Tensor product $- \boxtimes -\ =$ 
- Unit 1 $=$ 
- Associator $(-\boxtimes-)\boxtimes- \ \cong\ -\boxtimes(-\boxtimes-)$: a path 
- Braiding $-\boxtimes- \ \cong\ (-\boxtimes-)\circ \mathrm{flip}$: a path 

The (b) part can be very interesting even when the (a) part is trivial (e.g. Moonshine CFT). In genus zero, morally equivalent to vertex tensor category.

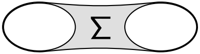
Examples

- Because the Segal CFT has so much data, it is very difficult to construct examples.
- Segal CFTs in this spirit has been constructed for lattice models (Posthuma '12) and the free fermion (T '19).
- Ongoing project with Henriques to construct *enhanced* Segal CFTs, where the cobordisms are allowed to be partially thin.
- This approach is advantageous because it allows access to the underlying conformal net, which characterizes the generators:



- The resulting structure 'sees' all aspects of the CFT: the conformal net and its representation category, the VOA, a vertex tensor category, mapping class group representations, and so on.

Open problems

- We expect to start with a conformal net $\mathcal{A}$ (such that the map Y_Σ associated to 

is continuous), and build an enhanced Segal CFT.

- In particular this produces a vertex tensor category of modules for the unitary VOA V associated to $\mathcal{A}$. Is there a VOA-theoretic characterization of these 'integrable' modules, e.g. when V is badly non-rational? What about the tensor product?

- Show that V is rational if and only if $\mathcal{A}$ is, and that the corresponding modular categories agree.

Thank you!